SUM RULES IN THE HEAVY QUARK LIMIT OF QCD AND ISGUR-WISE FUNCTIONS

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Using the OPE, we formulate new sum rules in the heavy quark limit of QCD. These sum rules imply that the elastic Isgur-Wise function $\xi(w)$ is an alternate series in powers of $(w-1)$. Moreover, one gets that the n-th derivative of $\xi(w)$ at $w=1$ can be bounded by the $(n-1)$-th one, and an absolute lower bound for the n-th derivative $(-1)^n\xi^{(n)}(1) \geq \frac{(2n+1)!!}{2^{2n}}$. Moreover, for the curvature we find $\xi''(1) \geq \frac{1}{5}[4\rho^2+3(\rho^2)^2]$ where $\rho^2 = -\xi'(1)$. We show that the quadratic term $\frac{3}{5}(\rho^2)^2$ has a transparent physical interpretation, as it is leading in a non-relativistic expansion in the mass of the light quark. These bounds should be taken into account in the parametrizations of $\xi(w)$ used to extract $|V_{cb}|$. These results are consistent with the dispersive bounds, and they strongly reduce the allowed region of the latter for $\xi(w)$. The method is extended to the subleading quantities in $1/m_Q$, namely $\xi_3(w)$ and $\overline{\Lambda}\xi(w)$.

In the leading order of the heavy quark expansion of QCD, Bjorken sum rule (SR) [1] relates the slope of the elastic Isgur-Wise (IW) function $\xi(w)$, to the IW functions of the transitions between the ground state and the $j^P = \frac{1}{2}^+$, $\frac{3}{2}^+$ excited states, $\tau_{1/2}^{(n)}(w)$, $\tau_{3/2}^{(n)}(w)$, at zero recoil $w=1$ (n is a radial quantum number). This SR leads to the lower bound $-\xi'(1) = \rho^2 \geq \frac{1}{4}$. Recently, a new SR was formulated by Uraltsev in the heavy quark limit [2] involving also $\tau_{1/2}^{(n)}(1)$, $\tau_{3/2}^{(n)}(1)$, that implies, combined with Bjorken SR, the much stronger lower bound $\rho^2 \geq \frac{3}{4}$, a result that came as a big surprise. In ref. [3], in order to make a systematic study in the heavy quark limit of QCD, we have developed a manifestly covariant formalism within the Operator Product Expansion (OPE). We did recover Uraltsev SR plus a new class of SR. Making a natural physical assumption, this new class of SR imply the bound $\sigma^2 \geq \frac{5}{4}\rho^2$ where σ^2 is the curvature of the IW function. Using this formalism including the whole tower of excited states j^P, we have recovered rigorously the bound $\sigma^2 \geq \frac{5}{4}\rho^2$ plus generalizations that extend it to all the derivatives of the IW function $\xi(w)$ at zero recoil, that is shown to be an alternate series in powers of $(w-1)$.

Using the OPE and the trace formalism in the heavy quark limit, different initial and final four-velocities v_i and v_f, and heavy quark currents, where Γ_1 and Γ_2 are arbitrary Dirac matrices $J_1 = \bar{h}_{v'}^{(c)}\ \Gamma_1\ h_{v_i}^{(b)}$, $J_2 = \bar{h}_{v_f}^{(b)}\ \Gamma_2\ h_{v'}^{(c)}$, the following sum rule can be written [4] :

$$\Big\{ \sum_{D=P,V} \sum_n Tr\left[\bar{\mathcal{B}}_f(v_f)\bar{\Gamma}_2\mathcal{D}^{(n)}(v')\right] Tr\left[\bar{\mathcal{D}}^{(n)}(v')\Gamma_1\mathcal{B}_i(v_i)\right]\xi^{(n)}(w_i)\xi^{(n)}(w_f) + \text{Other excited states}\Big\} = -2\xi(w_{if})\, Tr\left[\bar{\mathcal{B}}_f(v_f)\bar{\Gamma}_2 P'_+\Gamma_1\mathcal{B}_i(v_i)\right] \ . \qquad (1)$$

In this formula v' is the intermediate meson four-velocity, $P'_+ = \frac{1}{2}(1+\not{v}')$ comes from the residue of the positive energy part of the c-quark propagator, $\xi(w_{if})$ is the elastic Isgur-Wise function that appears because one assumes $v_i \neq v_f$. $\mathcal{B}_i$ and $\mathcal{B}_f$ are the 4×4 matrices of the ground state B or B^* mesons and $\mathcal{D}^{(n)}$ those of all possible ground state or excited state D mesons coupled to B_i and B_f

through the currents. In (1) we have made explicit the $j = \frac{1}{2}^-$ D and D^* mesons and their radial excitations of quantum number n. The explicit contribution of the other excited states is written below.

The variables w_i, w_f and w_{if} are defined as $w_i = v_i \cdot v'$, $w_f = v_f \cdot v'$, $w_{if} = v_i \cdot v_f$.

The domain of $(w_i,\ w_f,\ w_{if})$ is [3] $(w_i, w_f \geq 1)$

$$w_i w_f - \sqrt{(w_i^2-1)(w_f^2-1)} \leq w_{if}$$
$$\leq w_i w_f + \sqrt{(w_i^2-1)(w_f^2-1)}\ . \qquad (2)$$

The SR (1) writes $L\left(w_i, w_f, w_{if}\right) = R\left(w_i, w_f, w_{if}\right)$, where $L(w_i, w_f, w_{if})$ is the sum over the intermediate charmed states and $R(w_i, w_f, w_{if})$ is the OPE side. Within the domain (2) one can derive relatively to any of the variables w_i, w_f and w_{if} and obtain different SR taking different limits to the frontiers of the domain.

As in ref. [3], we choose as initial and final states the B meson $\mathcal{B}_i(v_i) = P_{i+}(-\gamma_5)$ $\mathcal{B}_f(v_f) = P_{f+}(-\gamma_5)$ and vector or axial currents projected along the v_i and v_f four-velocities

$$J_1 = \bar{h}_{v'}^{(c)}\ \not{v}_i\ h_{v_i}^{(b)} \quad , \qquad J_2 = \bar{h}_{v_f}^{(b)}\ \not{v}_f\ h_{v'}^{(c)} \quad (3)$$

we obtain SR (1) with the sum of all excited states j^P in a compact form :

$$(w_i+1)(w_f+1)\sum_{\ell\geq 0}\frac{\ell+1}{2\ell+1}S_\ell(w_i,w_f,w_{if})$$
$$\sum_n \tau_{\ell+1/2}^{(\ell)(n)}(w_i)\tau_{\ell+1/2}^{(\ell)(n)}(w_f)$$
$$+\sum_{\ell\geq 1}S_\ell(w_i,w_f,w_{if})\sum_n \tau_{\ell-1/2}^{(\ell)(n)}(w_i)\tau_{\ell-1/2}^{(\ell)(n)}(w_f)$$
$$= (1+w_i+w_f+w_{if})\xi(w_{if})\ . \qquad (4)$$

We get, choosing instead the axial currents,

$$J_1 = \bar{h}_{v'}^{(c)}\ \not{v}_i\ \gamma_5\ h_{v_i}^{(b)}\ ,\ J_2 = \bar{h}_{v_f}^{(b)}\ \not{v}_f\ \gamma_5\ h_{v'}^{(c)}\ , \qquad (5)$$

$$\sum_{\ell\geq 0}S_{\ell+1}(w_i,w_f,w_{if})$$
$$\sum_n \tau_{\ell+1/2}^{(\ell)(n)}(w_i)\tau_{\ell+1/2}^{(\ell)(n)}(w_f)$$
$$+(w_i-1)(w_f-1)$$
$$\sum_{\ell\geq 1}\frac{\ell}{2\ell-1}S_{\ell-1}(w_i,w_f,w_{if})$$
$$\sum_n \tau_{\ell-1/2}^{(\ell)(n)}(w_i)\tau_{\ell-1/2}^{(\ell)(n)}(w_f)$$
$$= -(1-w_i-w_f+w_{if})\xi(w_{if})\ . \qquad (6)$$

Following the formulation of heavy-light states for arbitrary j^P given by Falk [4], we have defined in ref. [3] the IW functions $\tau_{\ell+1/2}^{(\ell)(n)}(w)$ and $\tau_{\ell-1/2}^{(\ell)(n)}(w)$, ℓ and $j = \ell \pm \frac{1}{2}$ being the orbital and total angular momentum of the light quark.

In (3) and (5) S_n is given by

$$S_n = v_{i\nu_1}\cdots v_{i\nu_n}\ v_{f\mu_1}\cdots v_{f\mu_n}$$
$$\sum_\lambda \varepsilon'^{(\lambda)*\nu_1\cdots\nu_n}\ \varepsilon'^{(\lambda)\mu_1\cdots\mu_n}\ . \qquad (7)$$

One can show [3] :

$$S_n = \sum_{0\leq k\leq \frac{n}{2}} C_{n,k}(w_i^2-1)^k(w_f^2-1)^k$$
$$(w_i w_f - w_{if})^{n-2k} \qquad (8)$$

with $C_{n,k} = (-1)^k \frac{(n!)^2}{(2n)!}\ \frac{(2n-2k)!}{k!(n-k)!(n-2k)!}$.
From the sum of (4) and (6) one obtains, differentiating relatively to w_{if} [5] $(\ell \geq 0)$:

$$\xi^{(\ell)}(1) = \frac{1}{4}\,(-1)^\ell\,\ell!\left\{\frac{\ell+1}{2\ell+1}4\sum_n\left[\tau_{\ell+1/2}^{(\ell)(n)}(1)\right]^2\right.$$
$$\left.+\sum_n\left[\tau_{\ell-1/2}^{(\ell-1)(n)}(1)\right]^2+\sum_n\left[\tau_{\ell-1/2}^{(\ell)(n)}(1)\right]^2\right\}\ . \qquad (9)$$

This relation shows that $\xi(w)$ is an alternate series in powers of $(w-1)$. Equation (9) reduces to Bjorken SR [1] for $\ell = 1$. Differentiating (6) relatively to w_{if} and making $w_i = w_f = w_{if} = 1$ one obtains :

$$\xi^{(\ell)}(1) = \ell!\ (-1)^\ell\sum_n\left[\tau_{\ell+1/2}^{(\ell)(n)}(1)\right]^2 \quad (\ell\geq 0)\ . \qquad (10)$$

Combining (9) and (10) one obtains a SR for all ℓ that reduces to Uraltsev SR [2] for $\ell = 1$. From (9) and (10) one obtains :

$$(-1)^\ell\ \xi^{(\ell)}(1) = \frac{1}{4}\ \frac{2\ell+1}{\ell}\ell!$$

$$\left\{ \sum_n \left[\tau^{(\ell-1)(n)}_{\ell-1/2}(1) \right]^2 + \sum_n \left[\tau^{(\ell)(n)}_{\ell-1/2}(1) \right]^2 \right\}\ . \tag{11}$$

implying

$$(-1)^\ell \xi^{(\ell)}(1) \geq \frac{2\ell+1}{4} \left[(-1)^{\ell-1} \xi^{(\ell-1)}(1) \right]$$

$$\geq \frac{(2\ell+1)!!}{2^{2\ell}} \tag{12}$$

that gives, in particular, for the lower cases,

$$-\xi'(1) = \rho^2 \geq \frac{3}{4} \quad , \quad \xi''(1) \geq \frac{15}{16} \tag{13}$$

Considering systematically the derivatives of the SR (4) and (6) relatively to w_i, w_f, w_{if} with the boundary conditions $w_{if} = w_i = w_f = 1$, one obtains a new SR:

$$\frac{4}{3}\rho^2 + (\rho^2)^2 - \frac{5}{3}\sigma^2 + \sum_{n\neq 0} |\xi^{(n)'}(1)|^2 = 0 \tag{14}$$

that implies :

$$\sigma^2 \geq \frac{1}{5} \left[4\rho^2 + 3(\rho^2)^2 \right]\ . \tag{15}$$

There is a simple intuitive argument to understand the term $\frac{3}{5}(\rho^2)^2$ in the best bound (15), namely the non-relativistic quark model, i.e. a non-relativistic light quark q interacting with a heavy quark Q through a potential. The form factor has the simple form :

$$F(\mathbf{k}^2) = \int d\mathbf{r}\ \varphi_0^+(r)$$

$$\exp\left(i \frac{m_q}{m_q + m_Q} \mathbf{k}\cdot\mathbf{r} \right) \varphi_0(r) \tag{16}$$

where $\varphi_0(r)$ is the ground state radial wave function. Identifying the non-relativistic IW function $\xi_{NR}(w)$ with the form factor $F(\mathbf{k}^2)$ (16), one can prove that,

$$\sigma^2_{NR} \geq \frac{3}{5}\ \left[\rho^2_{NR} \right]^2\ . \tag{17}$$

Thus, the non-relativistic limit is a good guide-line to study the shape of the IW function $\xi(w)$. We have recently generalized the bound (17) to all the derivatives of $\xi_{NR}(w)$. The method uses the positivity of matrices of moments of the ground state wave function [6]. We have shown that the method can be generalized to the real function $\xi(w)$ of QCD.

An interesting phenomenological remark is that the simple parametrization for the IW function [7]

$$\xi(w) = \left(\frac{2}{w+1} \right)^{2\rho^2} \tag{18}$$

satisfies the inequalities (12), (15) if $\rho^2 \geq \frac{3}{4}$.

The result (12), that shows that all derivatives at zero recoil are large, should have important phenomenological implications for the empirical fit needed for the extraction of $|V_{cb}|$ in $B \to D^* \ell\nu$. The usual fits to extract $|V_{cb}|$ using a linear or linear plus quadratic dependence of $\xi(w)$ are not accurate enough. It is important to point out that the most precise data points are the ones at large w, so that higher derivatives contribute importantly in this region.

A considerable effort has been developed to formulate dispersive constraints on the shape of the form factors in $\bar{B} \to D^* \ell\nu$ [8]-[9], at finite mass.

Our approach, based on Bjorken-like SR, holds *in the physical region* of the semileptonic decays $\bar{B} \to D^{(*)} \ell\nu$ and *in the heavy quark limit*. The two approaches are quite different in spirit and in their results.

Let us consider the main results of ref. [9] summarized by the one-parameter formula

$$\xi(w) \cong 1 - 8\rho^2 z + (51\rho^2 - 10)z^2 - (252\rho^2 - 84)z^3 \tag{19}$$

with the variable $z(w)$ defined by

$$z = \frac{\sqrt{w+1} - \sqrt{2}}{\sqrt{w+1} + \sqrt{2}} \tag{20}$$

and the allowed range for ρ^2 being $-0.17 < \rho^2 < 1.51$. This domain is considerably tightened by the lower bound on ρ^2 : $\frac{3}{4} \leq \rho^2 <$

1.51, that shows that our type of bounds are complementary to the upper bounds obtained from dispersive methods.

By extension of our method to subleading order in $1/m_Q$, we have shown that the subleading quantities, that are functions of w, $\overline{\Lambda}\xi(w)$ and $\xi_3(w)$ can be expressed in terms of leading quantities, namely the $\frac{1}{2}^- \to \frac{1}{2}^+, \frac{3}{2}^+$ IW functions $\tau_j^{(n)}(w)$ and the corresponding level spacings $\Delta E_j^{(n)}$ $(j = \frac{1}{2}, \frac{3}{2})$ [10]

$$\overline{\Lambda}\xi(w) = 2(w+1)\sum_n \Delta E_{3/2}^{(n)}\ \tau_{3/2}^{(n)}(1)\ \tau_{3/2}^{(n)}(w) + 2\sum_n \Delta E_{1/2}^{(n)}\ \tau_{1/2}^{(n)}(1)\ \tau_{1/2}^{(n)}(w) \quad (21)$$

$$\xi_3(w) = (w+1)\sum_n \Delta E_{3/2}^{(n)}\ \tau_{3/2}^{(n)}(1)\ \tau_{3/2}^{(n)}(w) - 2\sum_n \Delta E_{1/2}^{(n)}\ \tau_{1/2}^{(n)}(1)\ \tau_{1/2}^{(n)}(w) \quad (22)$$

These quantities reduce to known SR for $w = 1$, respectively Voloshin SR [11] and a SR for $\xi_3(1)$ [12,2], and generalizes them for all w.

The comparison of (21), (22) with the results of the BT quark model [7] is very encouraging. Within this scheme $\xi(w)$ is given by (18) with $\rho^2 = 1.02$, while one gets, for the $n = 0$ states

$$\tau_j^{(0)}(w) = \tau_j^{(0)}(1)\left(\frac{2}{w+1}\right)^{2\sigma_j^2} \quad (23)$$

with $\tau_{3/2}^{(0)}(1) = 0.54$, $\sigma_{3/2}^2 = 1.50$, $\tau_{1/2}^{(0)}(1) = 0.22$ and $\sigma_{1/2}^2 = 0.83$. Assuming the reasonable saturation of the SR with the lowest $n = 0$ states [7], one gets, from the first relation (21), a sensibly constant value for $\overline{\Lambda} = 0.513 \pm 0.015$.

In conclusion, using sum rules in the heavy quark limit of QCD, as formulated in ref. [3,10], we have found lower bounds for the moduli of the derivatives of $\xi(w)$. Any phenomenological parametrization of $\xi(w)$ intending to fit the CKM matrix element $|V_{cb}|$ in $B \to D^{(*)}\ell\nu$ should satisfy these bounds. Moreover, we discuss these bounds in comparison with the dispersive approach. We show that there is no contradiction, our bounds restraining the region for $\xi(w)$ allowed by this latter method. Moreover, we have found non-trivial new information on subleading contributions in $1/m_Q$.

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